Achieving a decision in antagonistic multiagent networks: frustration determines commitment strength*

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Abstract— **In this work we consider a nonlinear interconnected system describing a decision-making process in a community of agents characterized by the coexistence of collaborative and antagonistic interactions. The resulting signed graph is in general not structurally balanced. It is shown in the paper that the decision-making process is affected by the frustration of the signed graph, in the sense that a nontrivial decision can be reached only if the social commitment of the agents is high enough to win the disorder introduced by the frustration in the network. The higher the frustration of the graph, the higher the commitment strength required from the agents.**

I. Introduction

Models of interconnected systems are widely used to gain insight into the complex dynamics of groups of agents and into their emerging behavior. For instance, in opinion dynamics, each agent can represent an individual of a "social network", and it is itself represented as a node in a graph. The edges of the network correspond in this case to the social ties between individuals, and form an adjacency matrix sometimes referred to as "sociomatrix" [1]. When the social ties are all of collaborative type, then it is natural to consider an adjacency matrix which is nonnegative. When instead collaborative and antagonistic interactions coexist, then it is convenient to consider a signed adjacency matrix. In this case also the corresponding graph is a signed graph. Signed graphs have been studied for a long time in social network theory [1], [2]. For instance, the notion of structural balance which was introduced by F. Harary in [3], [4], captures the graph-theoretical meaning of common sense notions like "the friend of my enemy is my enemy", and extends them to arbitrary graphs. It states that a graph is structurally balanced if all its cycles are positive, i.e., are formed by an even number of negative edges. In particular, if one considers the variant of the Laplacian matrix associated to a signed graph [5], here called signed Laplacian, then structural balance of a signed graph corresponds to the signed Laplacian having 0 among its eigenvalues.

When a connected signed graph is not structurally balanced, then its signed Laplacian does not have 0 as an eigenvalue and it becomes of interest to give an estimation of the distance of the graph from the balanced state. For the case of unweighted graphs, the literature offers several measures of graph imbalance based on different properties of the signed graphs, such as cycles, eigenvalues, frustration, see [6] for an overview. In particular, it is common to consider the frustration index (or line index of imbalance [7]), i.e., the minimum number of edges whose deletion makes the signed graph structurally balanced [8], [9], [10]. The actual computation of this value is a NP-hard problem, and several algorithms have been proposed, see for example [11] which is based on the idea that the frustration index is equal to the minimum number of negative fundamental cycles of the graph, or [12] where the analogy with the statistical physics problem of minimizing the energy of an Ising spin glass is used. For small-medium size networks, [9] shows how to compute an exact value of the frustration index as solution of an optimization problem. Given the complexity of the problem, upper and lower bounds for the frustration index have been introduced in the literature, based on the properties of the graph [13], [14], or related to the smallest strictly positive eigenvalue of the signed unweighted Laplacian [8], [15], or of the normalized signed unweighted Laplacian [16]. In this paper, such notions are extended to weighted signed graphs, see also [17]. In particular, we show that the weighted frustration index has a behavior similar to its unweighted counterpart, provided we look at normalized Laplacians.

If we consider a multiagent dynamical system on a signed network, then it makes sense to identify the sign of an edge with the sign of the corresponding entry of the Jacobian matrix of the dynamics: friends exert a positive influence and "enemies" a negative one. We assume that this sign identification is valid everywhere, not just around an equilibrium point. The models proposed in e.g. [18], [19], [20] serve well for this scope and will be adopted also in this paper. They essentially use sigmoidal nonlinearities, multiplied by the edge weights, to describe the influences of an agent on the other agents [21], [22], and resemble closely interconnected models used to describe animal group behavior [23], [19] or neural networks [24], [25].

Similarly to [19], [20], we choose the weights so as to get a linearization which is a Laplacian matrix, and we endow our multiagent model with a scalar parameter representing the "social effort" of the group of agents, playing the role of bifurcation parameter. For low values of this parameter, the linearization at the origin is globally asymptotically stable, meaning that the commitment of the agents is not enough to achieve a nontrivial decision. As the social effort grows, the system experiences a pitchfork bifurcation, with the origin becoming unstable and two nontrivial locally asymptotically stable equilibria appearing, playing the role of alternative decision states for the community. However, unlike for the cooperative systems studied in [19], [20], for our signed

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graphs the value of social effort at which the bifurcation is crossed is not fixed and constant, but it depends on the frustration index of the signed graph. In particular we show in the paper that the frustration index of a structurally unbalanced graph is (almost) proportional to the smallest eigenvalue of the normalized signed Laplacian, which in turn tells us that the social effort needed to achieve a nontrivial decision grows with the frustration index. In other words, in order to achieve a nontrivial collective decision, a social network with frustration requires a higher commitment of its individuals than a structurally balanced social network.

In the structurally balance case, our system is monotone [26] and its behavior is identical to that of the cooperative system analyzed in [19], [20], modulo a change of orthant. This means that on each decision state two factions will form, having opposite opinions. Also in our structurally unbalanced graphs the decision states that appear for high enough values of social commitment are of mixed sign, i.e., two factions emerge even in absence of structural balance.

Like in the cooperative case, a second threshold on the value of the social commitment is possible, beyond which a second bifurcation happens, meaning that more equilibria (i.e., decision states) can appear. Similarly to [20], we show that this second threshold is related to the algebraic connectivity of the signed graph. Unlike the smallest eigenvalue of the normalized signed Laplacian, the algebraic connectivity is basically insensitive to frustration. Hence, as the frustration grows, the gap between first and second eigenvalue of the normalized signed Laplacian shrinks, meaning that the interval of values in which only two competing equilibria coexist also shrinks. Interpreted as a robustness property, this result says that nonzero decision states become less robust to perturbation (of topology, edge weight, edge or node loss, etc.) as the frustration grows.

II. SIGNED GRAPHS AND FRUSTRATION

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$ and edge set $\mathcal{E} = \{e_1, \ldots, e_m\}$, and let $n = |\mathcal{V}|$ be the number of nodes in the graph. In this work we will consider undirected graphs, for which $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$, without loops and multiple edges. Associated with the graph $\mathcal{G}$ is the adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ in which $a_{ij} \neq 0$ iff $(j, i) \in \mathcal{E}$; $\mathcal{G}$ undirected and without loops implies that the matrix A is symmetric with null-diagonal. A graph $\mathcal{G}$ is *signed* if each of its edges is labeled by a sign, $\text{sign}\,(e_i) = \pm 1$ for all $i = 1, \ldots, m$, which translates in $\text{sign}\,(a_{ij}) = \text{sign}\,(a_{ji}) = \pm 1$ if $(i, j) \in \mathcal{E}$.

The *signed Laplacian* of a graph $\mathcal{G}$ is the symmetric matrix $L = \Delta - A$, where Δ is a diagonal matrix whose elements are given by $\delta_i = \sum_{j=1}^{n} |a_{ij}|$ for all i. If the graph $\mathcal{G}$ is unsigned, this definition equals the standard Laplacian matrix.

We further assume that the graph is *connected*, i.e., there exists a path from a node v_i to a different node v_j for every $v_i, v_j \in \mathcal{V}$, property that translates into the adjacency matrix A being irreducible. As a consequence, $\mathcal{G}$ does not have isolated vertices, i.e., $\delta_i \neq 0$ for all i. Hence, the matrix Δ^{-1} is well-defined and positive definite.

The *normalized signed Laplacian* of a graph $\mathcal{G}$, see [17], is the non-symmetric (symmetrizable [21]) matrix defined as $\mathcal{L} = \Delta^{-1} L = I - \Delta^{-1} A$.

A cycle of a signed graph $\mathcal{G}$ is said *positive* if it contains an even number of negative edges, *negative* otherwise [11]. A graph is *structurally balanced* if all its cycles are positive.

A. Frustration index and the smallest Laplacian eigenvalue

The following theorem is a well-known result in the literature, see for example [5], [27].

Theorem 1 *Let $\mathcal{G}$ be a connected signed graph. Then the following conditions are equivalent:*

1) *$\mathcal{G}$ is structurally balanced;*
2) *There exists a signature matrix $S = diag\,\{s_1, \ldots, s_n\}$ with diagonal entries $s_i = \pm 1$, such that $S\mathcal{L}S$ has all nonpositive off-diagonal entries;*
3) *$\lambda_1(\mathcal{L}) = 0$.*

For unweighted graphs a measure of graph imbalance is given by the *frustration index*, while other works instead consider the *algebraic conflict*, i.e., the least Laplacian eigenvalue [6], [28]. These measures are strictly related, as shown by the bounds introduced in the literature [8], [15], [16]. Analogous definitions can be given for weighted graphs.

Definition 1 (Frustration index) *The weighted frustration index of a signed graph $\mathcal{G}$, denoted $\epsilon(\mathcal{G})$, is the minimum weighted sum of the positive edges over all signature similarity transformations of $\mathcal{L}$, $\mathcal{L}_s = S\mathcal{L}S$ with S a signature matrix:*

$$\epsilon(\mathcal{G}) = \min_{\substack{S=diag\{s_1,\ldots,s_n\},\\ s_i=\pm 1}} \quad \frac{1}{2} \sum_{i\neq j} \left([\mathcal{L}_s]_{ij} + [|\mathcal{L}_s|]_{ij} \right). \qquad (1)$$

From Theorem 1 if $\mathcal{G}$ is structurally balanced, then $\epsilon(\mathcal{G}) = 0$. When $\mathcal{G}$ is not balanced, then $\epsilon(\mathcal{G})$ is the least amount of edge weights whose deletion (or sign change) makes the graph balanced over all possible similarity transformations with S.

Definition 2 (Algebraic conflict) *The algebraic conflict of a signed graph $\mathcal{G}$, denoted $\xi(\mathcal{G})$, is the smallest eigenvalue of the normalized signed Laplacian $\mathcal{L}$ of $\mathcal{G}$: $\xi(\mathcal{G}) = \lambda_1(\mathcal{L})$.*

Both $\epsilon(\mathcal{G})$ and $\xi(\mathcal{G})$ can be used to characterize the graph distance from the structurally balanced state. In Example 1 we show that the two are correlated, although not identical.

B. The second smallest Laplacian eigenvalue

For an unsigned graph $\mathcal{G}$, the value of the second smallest eigenvalue of the normalized Laplacian, denoted *algebraic connectivity*, is related to the connectivity of $\mathcal{G}$ and, in particular, $\lambda_2(\mathcal{L}) = 0$ if and only if $\mathcal{G}$ is not connected [29].

Through numerical analysis on Erdős-Rényi networks, see Example 1, it is possible to observe that while $\lambda_1(\mathcal{L})$ changes with β, i.e., with different signatures over the edges, the value of $\lambda_2(\mathcal{L})$ remains basically constant, implying that the maximum distance $\lambda_2(\mathcal{L}) - \lambda_1(\mathcal{L})$ is obtained for a structurally balanced (or unsigned) graph. This is the situation described in [20].

III. Decision-making in antagonistic nonlinear multiagent systems

Consider the following class of nonlinear interconnected systems

$$\dot{x} = f(x, \pi) = -\Delta x + \pi A \psi(x), \quad x \in \mathbb{R}^n \tag{2}$$

where $\Delta = \operatorname{diag}\{\delta_1, \dots, \delta_n\}$, $A = [a_{ij}]$ is the adjacency matrix of the network $\mathcal{G}$, $\pi > 0$ is a positive scalar parameter, and $\psi(x) = [\psi_1(x_1) \dots \psi_n(x_n)]^T$. In the context of social networks, the entry x_i of the state vector $x = [x_1 \dots x_n]^T \in \mathbb{R}^n$ represents the opinion of the i-th agent, and the matrix A describes the interactions among the agents: for each element a_{ji}, its sign defines the relationship among the agents i and j, friendly $(+)$ or unfriendly $(-)$, while its absolute value is the amount of "trust" (or "distrust") that agent i puts on agent j. When the matrix is symmetric, this translates into agents i and j sharing the same amount of trust/distrust. The function $\psi_i(x_i)$ describes how an agent i expresses its opinion to its first neighbors. The same $\psi_i(x_i)$ is "broadcasted" to all neighbors, weighted by the a_{ji} terms. The parameter π represents the *social effort*, or *strength of the commitment* among the agents, and indicates the collective amount of commitment to the overall interaction process [19].

We assume that the weighted matrix A is signed with null diagonal, irreducible and symmetric; moreover we assume that a Laplacian-like assumption relates Δ and A, $\delta_i = \sum_j |a_{ij}|$ for all i. Finally, let each function $\psi_i(x_i) : \mathbb{R} \to \mathbb{R}$ of the vector $\psi(x)$ satisfy the following conditions:

$$\psi_i(x_i) = -\psi_i(-x_i), \forall x_i \in \mathbb{R} \quad \text{(odd)} \tag{A.1}$$

$$\frac{\partial \psi_i}{\partial x_i}(x_i) > 0, \forall x_i \in \mathbb{R} \text{ and } \frac{\partial \psi_i}{\partial x_i}(0) = 1 \text{ (monotone)} \tag{A.2}$$

$$\lim_{x_i \to \pm\infty} \psi_i(x_i) = \pm 1 \quad \text{(saturated)} \tag{A.3}$$

$$\psi_i(x_i) \begin{cases} \text{strictly convex} & \forall\, x_i < 0 \\ \text{strictly concave} & \forall\, x_i > 0 \end{cases} \quad \text{(sigmoidal).} \tag{A.4}$$

At the origin, when $\pi = 1$, the Jacobian is $J = \frac{\partial f}{\partial x}(0, 1) = -(\Delta - A) = -L$. The system (2) is *monotone* if the Kamke condition is satisfied, see [26], which in graph theory translates into $\mathcal{G}$ being structurally balanced.

Our task is to investigate the presence and the stability of equilibrium points of the system (2) with respect to the bifurcation parameter π. The system (2) can be rewritten in a "normalized" form,

$$\dot{x} = \Delta \left[-x + \pi H \psi(x) \right], \quad x \in \mathbb{R}^n, \tag{3}$$

where $H := \Delta^{-1} A$. Note that H is symmetrizable, therefore its eigenvalues are real [21]; let them be arranged in a nondecreasing order. Since H is signed, it follows that $\lambda_n(H) \leq \rho(H) \leq \rho(|H|) = 1$ [30].

A. Previous results: structurally balanced graphs

If the system (3) is cooperative, its adjacency matrix is nonnegative, meaning that only friendly relationships exist among the agents. In this case it is known how the presence of equilibria depends on the value of the bifurcation parameter π, see [20], [31]. The same result holds in general when the system is monotone, i.e., when the graph is structurally balanced. The main results we obtained in [20] are summarized in the following theorem.

Theorem 2 *Consider the system* (3) *where each nonlinear function* $\psi_i(x_i)$ *satisfies the properties* (A.1)$\div$(A.4). *Assume that the signed graph* $\mathcal{G}$ *is structurally balanced and let* S *be the signature matrix s.t.* $S\mathcal{L}S$ *has all nonpositive off-diagonal entries* ($|A| = SAS$)*.*

- *When* $\pi < 1$ *the origin is the unique equilibrium point and it is asymptotically stable.*
- *When* $\pi = \pi_1 = 1$ *the system undergoes a pitchfork bifurcation, the origin becomes unstable and two new equilibria appear in the orthants described by* S *and* $-S$*, respectively, denoted* $S\mathbb{R}^n_+$ *and* $S\mathbb{R}^n_-$*. These equilibria are locally asymptotically stable with domain of attraction at least equal to* $S\mathbb{R}^n_+$ *and* $S\mathbb{R}^n_-$*, respectively.*
- *If* $\lambda_2(\mathcal{L}) < 1$ *and simple, when* $\pi = \pi_2 = \frac{1}{1 - \lambda_2(\mathcal{L})}$ *the system undergoes a second pitchfork bifurcation, and new equilibria in other orthants of* $\mathbb{R}^n$ *appear, which may be stable or unstable.*

B. New results: structurally unbalanced graphs

Our task is to extend the analysis carried out in [19], [20] to the context of general signed social networks. We therefore assume that the matrix A is signed. The following theorem presents the necessary and sufficient condition for the system (3) to admit (at least) one equilibrium point in a generic orthant of $\mathbb{R}^n$. Similarly to [20], the necessary part of the proof relies on geometric considerations, while the sufficient part uses singularity analysis of bifurcations.

Theorem 3 *Consider the system* (3) *where each nonlinear function* $\psi_i(x_i)$ *satisfies the properties* (A.1)$\div$(A.4). *Assume that the largest eigenvalue of the normalized interaction matrix* H*,* $\lambda_n(H)$*, is simple. The system admits an equilibrium point* $x^* \neq 0$ ($|x^*| > 0$)*, if and only if* $\pi > \pi_1 = \frac{1}{\lambda_n(H)}$.

Proof.

[Necessity] Let $x^* \neq 0$ ($|x^*| > 0$) be an equilibrium point for the system (3), which is $x^* = \pi H \psi(x^*)$. Let $M = \operatorname{diag}\{m_1, \dots, m_n\}$ with $m_i = \frac{\psi_i(x_i^*)}{x_i^*} \in (0, 1)\ \forall\ i \in \mathcal{I} := \{1, \dots, n\}$, which follows from (A.2) and (A.4). Then

$$x^* = \pi H \psi(x^*) = \pi H M x^* \tag{4}$$

that is, $(\frac{1}{\pi}, x^*)$ is an eigenpair of HM. Let $H_{\text{sym}} = \Delta^{\frac{1}{2}} H \Delta^{-\frac{1}{2}} \sim H$ be the symmetric version of H, where $\sim$ indicates similarity, and apply the change of coordinates $z^* = (M\Delta)^{\frac{1}{2}} x^*$ to (4). Then $z^* = \pi M^{\frac{1}{2}} H_{\text{sym}} M^{\frac{1}{2}} z^*$ implies that $(\frac{1}{\pi}, z^*)$ is an eigenpair of $M^{\frac{1}{2}} H_{\text{sym}} M^{\frac{1}{2}}$ which is now a symmetric matrix: $\exists\, k \in \mathcal{I}$ such that $\lambda_k(M^{\frac{1}{2}} H_{\text{sym}} M^{\frac{1}{2}}) = \frac{1}{\pi}$. By Ostrowski's Theorem (Theorem 4.5.9 in [30]) it follows that there exists a $\theta_k \in [\min_i\{m_i\}, \max_i\{m_i\}]$ such that $\theta_k \lambda_k(H_{\text{sym}}) = \lambda_k(M^{\frac{1}{2}} H_{\text{sym}} M^{\frac{1}{2}}) = \frac{1}{\pi}$. The condition $0 < \theta_k \leq \max_i\{m_i\} < 1$ implies that H_{sym} has a positive eigenvalue $\lambda_k(H_{\text{sym}}) > \frac{1}{\pi}$, for a certain $k \in \mathcal{I}$.

Since H is similar to $H_{\rm sym}$, it follows that also H has a positive eigenvalue $\lambda_k(H)$ s.t. $1 < \pi\lambda_k(H)$. Finally, let $\pi_1 = \frac{1}{\lambda_n(H)}$; since $\lambda_k(H) \le \lambda_n(H)$ for all $k \in \mathcal{I}$, we obtain $1 < \pi\lambda_k(H) \le \pi\lambda_n(H) = \frac{\pi}{\pi_1}$. Therefore the system (3) admits a nontrivial equilibrium point x^* only if $\pi > \pi_1$.
[Sufficiency] To prove the sufficiency part of the theorem we use bifurcation theory as explained in [33]. The aim is to study the change in the number of solutions of

$$\Phi(x,\pi) = -x + \pi H\psi(x) = 0 \tag{5}$$

when the parameter π is varying. Let $J = \frac{\partial \Phi}{\partial x}(0,\pi_1) = -I + \pi_1 H$. If $\lambda_n(H)$ is a simple eigenvalue of H, rank $J = n-1$; let v and w be the right and left eigenvectors associated to $\lambda_n(H)$, normalized such that $w^T v = 1$. Following the Liapunov-Schmidt reduction approach, let E denote the projection of $\mathbb{R}^n$ onto $\mathrm{range}(J) = \big(\ker(J^T)\big)^{\perp} = (\mathrm{span}\{w\})^{\perp}$, $E = I - vw^T$, and $I - E$ the projection onto $\ker(J) = \mathrm{span}\{v\}$. Then the system of equations (5) near $(0,\pi_1)$ can be expanded into the equivalent system

$$E\,\Phi(x,\pi) = E\,(-x + \pi H\psi(x)) = 0 \tag{6a}$$
$$(I-E)\,\Phi(x,\pi) = (I-E)\,(-x + \pi H\psi(x)) = 0. \tag{6b}$$

Decompose x in the form $x = y\,v + r$, where $y \in \mathbb{R}$ and $r = E\,x \in (\mathrm{span}\{w\})^{\perp}$. Given that $\lambda_n(H)$ is simple, it follows that (6a) can be solved for $n-1$ of the x variables, $r = R(yv,\pi)$, applying the implicit function theorem. Substituting the solution r of (6a) in (6b), one obtains the reduced mapping ϕ, or center manifold,

$$\begin{aligned}\phi(y,\pi) = (I-E)\big[-yv - R(yv,\pi)\\ + \pi H\psi\big(yv + R(yv,\pi)\big)\big] = 0.\end{aligned} \tag{7}$$

Defining $g(y,\pi) = w^T\phi(y,\pi)$, its zeros are in one-to-one correspondence with the solutions of (5), and if the partial derivatives g_y, g_{yy}, g_{yyy}, g_π, $g_{\pi y}$ at $(0,\pi_1)$ satisfy

$$g = g_y = g_{yy} = g_\pi = 0,\;\; g_{yyy}\,g_{\pi y} < 0 \tag{8}$$

then (8) solves the recognition problem for a pitchfork bifurcation [33]. Using the notation introduced in [33] (in particular equation (3.23), which shows how the explicit expression of r is not needed), and observing that Φ is odd, the calculations reduce to

$$\begin{aligned}\Phi_y(0,\pi_1) = \frac{\partial \Phi}{\partial x}(x,\pi)\Big|_{(0,\pi_1)} v = \left(-I + \pi H\frac{\partial\psi(x)}{\partial x}\right)\Big|_{(0,\,\pi_1)} v\\ = (-I + \pi_1 H)\,v = \big(-1 + \pi_1\lambda_n(H)\big)\,v = 0\end{aligned} \tag{9}$$

$$\Phi_{yy}(0,\pi_1) = \pi H\left(\frac{\partial}{\partial x}\left(\frac{\partial\psi(x)}{\partial x}v\right)\right)\Big|_{(0,\pi_1)} v = 0 \tag{10}$$

$$\Phi_\pi(0,\pi_1) = \frac{\partial\Phi}{\partial\pi}(x,\pi)\Big|_{(0,\pi_1)} = H\psi(x)|_{(0,\pi_1)} = 0, \tag{11}$$

where we have used the assumptions (A.1), (A.2) and (A.4), giving $\psi(0) = 0$, $\frac{\partial\psi(x)}{\partial x}(0) = I$, and $\frac{\partial^2\psi_i}{\partial x_i^2}(0) = 0\ \forall i$. Therefore, (9), (10) and (11) yield $g_y(0,\pi_1) = g_{yy}(0,\pi_1) = g_\pi(0,\pi_1) = 0$. The two remaining partial derivatives are

$$\Phi_{\pi y}(0,\pi_1) = H\,\frac{\partial\psi(x)}{\partial x}\Big|_{(0,\pi_1)} v = Hv = \lambda_n(H)\,v$$

$$\begin{aligned}\Phi_{yyy}(0,\pi_1) &= \pi H\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial\psi(x)}{\partial x}\,v\right)v\right)\right)\Big|_{(0,\pi_1)} v\\ &= \pi_1 H\,\frac{\partial^3\psi}{\partial x^3}(0)\begin{bmatrix} v_1^3\\ \vdots\\ v_n^3\end{bmatrix}\end{aligned}$$

where we denote $\frac{\partial^3\psi}{\partial x^3}(0) := \mathrm{diag}\left\{\frac{\partial^3\psi_1}{\partial x_1^3}(0),\ldots,\frac{\partial^3\psi_n}{\partial x_n^3}(0)\right\}$. From (A.4) it follows that $\frac{\partial^3\psi}{\partial x^3}(0)$ is negative definite. Therefore, $g_{\pi y}(0,\pi_1) = w^T\Phi_{\pi y}(0,\pi_1) = \lambda_n(H) > 0$ and $g_{yyy}(0,\pi_1) = w^T\Phi_{yyy}(0,\pi_1) = \sum_{i=1}^n \frac{\partial^3\psi_i}{\partial x_i^3}(0) w_i v_i^3 < 0$, since $w_i v_i^3 \ge 0\ \forall i$. Then (8) holds, which solves the recognition problem for a pitchfork bifurcation, meaning that the number of solutions "jumps" from one to three. Hence at $\pi = \pi_1$ the system crosses a bifurcation through the origin and two new equilibria appear along $\mathrm{span}\{v\}$. ■

Corollary 1 *If the system* (3) *admits an equilibrium point* $x^* \neq 0$*, then also* $-x^*$ *is an equilibrium point.*

Proof. It follows from the assumption (A.1). Let x^* be an equilibrium point. Then $-x^* = -\pi H\psi(x^*) = \pi H\psi(-x^*)$. ■

Remark 1 The value $\pi_1 = \frac{1}{\lambda_n(H)}$ is positive and well-defined. In fact, assume that $\lambda_n(H) \le 0$ which, together with $0 = \mathrm{Tr}(H) = \sum_{i=1}^n \lambda_i(H)$, implies $\lambda_i(H) = 0$ for all $i = 1,\ldots,n$. Given that Δ is positive definite, it follows that $\lambda_i(A) = 0$ for all i; however, A nilpotent and symmetric implies A zero matrix, which is a contradiction.

Remark 2 The condition "$\lambda_n(H)$ is simple" is not always satisfied for a signed structurally unbalanced graph $\mathcal{G}$, see [32]. In this case the intuition, suggested by the reading of [33] and by numerical analysis, is that multiple equilibria (more than three) for the system (3) arise when $\pi > \pi_1$.

Theorem 3 shows that the system undergoes a pitchfork bifurcation when $\pi = \pi_1 = \frac{1}{\lambda_n(H)} = \frac{1}{1-\lambda_1(\mathcal{L})}$, by definition of normalized signed Laplacian. Next, we prove the instability of the origin and the asymptotic stability of the two new equilibria of the system, using the notation introduced in the proof of Theorem 3, in particular $\Phi(x,\pi) = -x + \pi H\psi(x)$.

Corollary 2 (Stability) *Under the assumptions of Theorem 3, when* $\pi = \pi_1$ *the system undergoes a pitchfork bifurcation, the origin becomes unstable and two new equilibria appear, which are locally asymptotically stable.*

Proof. The existence is shown in the proof of Theorem 3, where it is also proven that the bifurcation is a pitchfork.

The linearized system at the origin is $\dot{x} = \Delta\frac{\partial\Phi}{\partial x}(0,\pi)x = \Delta(-I + \pi H)x$, where Δ is positive definite and $\frac{\partial\Phi}{\partial x}(0,\pi)$

has eigenvalues $\pi\lambda_1(H)-1, \ldots, \pi\lambda_n(H)-1$. When $\pi > \pi_1$, $\frac{\partial \Phi}{\partial x}(0, \pi)$ has at least one positive eigenvalue, which proves the instability of the origin as equilibrium point of (3).

To prove that instead the new equilibrium point x^* (and $-x^*$) is (locally) asymptotically stable, we use Theorem I4.1 [33]. Let $\dot{z} = Jz$ be the linearization of $\dot{x} = \Phi(x, \pi)$ at $(0, \pi_1)$, where $J = \frac{\partial \Phi}{\partial x}(0, \pi_1) = -I + \pi_1 H$ has one zero eigenvalue and $n-1$ strictly negative eigenvalues. Then x^*, solution of $\Phi(x, \pi) = 0$ and corresponding to the solution y^* of $g(y, \pi) = 0$ (defined in the proof of Theorem 3), is asymptotically stable if $g_y(y^*, \pi) < 0$. Theorem 3 shows that in a neighborhood of the bifurcation $x^* \in \text{span}\{v\}$, where v is the right eigenvector associated with $\lambda_n(H)$. Moreover $x^* = y^* v = \pi H \psi(y^* v)$ implies $y^* = \pi\lambda_n(H) w^T \psi(y^* v)$ (since $w^T v = 1$). Since $\psi(\cdot)$ satisfies assumption (A.4), then

$$\begin{aligned} g_y(y^*, \pi) &= -1 + \pi\lambda_n(H)\, w^T \frac{\partial \psi}{\partial x}(y^* v)\, v \\ &< -1 + \frac{\pi\lambda_n(H) w^T \psi(y^* v)}{y^*} = 0. \end{aligned} \qquad \blacksquare$$

Theorem 4 *Under the assumptions of Theorem 3, if $\lambda_2(\mathcal{L}) < 1$ and simple, when $\pi = \pi_2 = \frac{1}{1-\lambda_2(\mathcal{L})}$ the system* (3) *undergoes a second pitchfork bifurcation, and new equilibria in other orthants of $\mathbb{R}^n$ appear.*

The proof, which follows from arguments similar to those introduced in the proof of Theorem 3, is omitted for lack of space.

Remark 3 The equilibria mentioned in Theorem 4 are unstable at the bifurcation, since the linearized system at $(0, \pi_2)$ described by $J = -I + \pi_2 H$ is unstable. However, when the value of π increases further, the system can bifurcate again, leading to new equilibria which can be stable or unstable.

IV. Interpretation of the results

Theorems 3 and 4, as well as the numerical analysis shown in the next section, suggest that the behavior described in [20], [31] still hold in the more generic case of a signed graph $\mathcal{G}$. Indeed, if $\lambda_1(\mathcal{L})$ is simple and $\lambda_2(\mathcal{L}) < 1$, it is possible to define a value $\pi_2 > \pi_1$ and hence an interval (π_1, π_2) of values of π for which the system (3) admits, in addition to the origin, only two alternative equilibrium points x^* and $-x^*$ which are locally asymptotically stable. As shown in Theorem 3, the system (3) undergoes a first pitchfork bifurcation for a value of the social effort that depends on the smallest eigenvalue of the normalized signed Laplacian. Since the algebraic conflict $\xi(\mathcal{G}) = \lambda_1(\mathcal{L})$ represents a measure of the structural imbalance of $\mathcal{G}$, it can be observed that having a highly unbalanced graph implies that the system (3) needs a higher social effort in order to converge to some equilibrium point x^* different from the origin. This equilibrium point represents a decision among the agents, that could be of agreement (if $x^* \in \mathbb{R}^n_+, \mathbb{R}^n_-$) or disagreement (otherwise), which does not depend on the initial opinions.

If we assume that $\lambda_1(\mathcal{L})$ is simple, the system bifurcates a second time when $\pi > \pi_2$, where π_2 depends on the second smallest eigenvalue of the normalized signed Laplacian, the algebraic connectivity. In this case, the system (3) admits multiple equilibria which can be stable or unstable and the convergence to an equilibrium point is determined by the initial conditions, see Example 2. These and other properties of the system are illustrated in the examples of the next section. The main features can be summarized as follows.

- The social effort needed to achieve a nontrivial decision is (roughly) proportional to the frustration index $\epsilon(\mathcal{G})$, since $\xi(\mathcal{G}) = \lambda_1(\mathcal{L}) = 1 - \lambda_n(H)$ and $\epsilon(\mathcal{G}) \sim \xi(\mathcal{G})$.
- The "robustness" of the decision, i.e., the interval (π_1, π_2), decreases with the increase of the frustration.
- The amount of frustration that can be encoded in a graph depends on the algebraic topology of the graph, i.e., $\lambda_1(\mathcal{L})$ is upper bounded by $\lambda_2(\mathcal{L})$.

V. Examples

Example 1 In this example we consider a sequence of signed weighted graphs, whose signature is dependent on a parameter $\beta \in [0, 1]$. Each graph $\mathcal{G}$ of the sequence is an Erdős-Rényi graph with $n = 500$ nodes, edge probability $p = 0.4$, and for which the value of β defines the negative edge probability as follows: $P[\text{negative edge in } \mathcal{G}] = p\beta$. The sequence is considered for increasing values of β, equal to $0, 0.05, \ldots, 0.95, 1$, hence the number of negative edges of each graph $\mathcal{G}$ grows in a linear way with β.

Our first aim is to show that the algebraic conflict gives a measure of imbalance of the graph, as it behaves similarly to the frustration index; in Fig. 1(a) it is possible to notice that the frustration index $\epsilon(\mathcal{G})$ grows linearly with the algebraic conflict $\lambda_1(\mathcal{L})$. However, when $\beta \geq 0.5$, $\epsilon(\mathcal{G})$ remains (roughly) constant while $\lambda_1(\mathcal{L})$ slightly increases. As for unweighted graphs, the intuition is that the algebraic conflict is an approximation of the frustration index and that it could be used to bound its value. The choice of using the smallest normalized Laplacian eigenvalue, see Fig. 1(b), instead of the smallest Laplacian eigenvalue, see Fig. 1(c), as measure of algebraic conflict is justified by the former resembling more closely what happens on unweighted graphs [6] and by our need to use the eigenvalues of the normalized Laplacian in Theorems 3 and 4 (see [20] for further details).

The second task of this example is to investigate the size of the gap between the two smallest eigenvalues of the normalized signed Laplacian when only β, i.e., the percentage of negative edges in the graph, changes. As depicted in Fig. 1(d), the gap is maximum when the graph is unsigned (or structurally balanced). At around $\beta = 0.5$, the gap is basically closed ($\lambda_2(\mathcal{L}) - \lambda_1(\mathcal{L}) = 0.002$) meaning that the two pitchfork bifurcations happen almost simultaneously. Hence the strong social commitment value (i.e. π) needed to achieve a decision in this case results in multiple (more than 3) possible decision states.

Example 2 Consider an Erdős-Rényi graph with $n = 50$ nodes, edge probability $p = 0.4$, and negative edge probability $p\beta$, $\beta = 0.4$. Both $\lambda_1(\mathcal{L})$ and $\lambda_2(\mathcal{L})$ are simple, and $\pi_1 = 1.97$, while $\pi_2 = 2.24$. In Figure 2, we show that when $\pi < \pi_1$, the origin is the only equilibrium point (black dots). When $\pi \in (\pi_1, \pi_2)$, the system (3) admits only x^* and $-x^*$ as (locally) asymptotically stable equilibrium points (red dots). When $\pi > \pi_2$, new equilibria arise, which can be stable or unstable (blue dots).

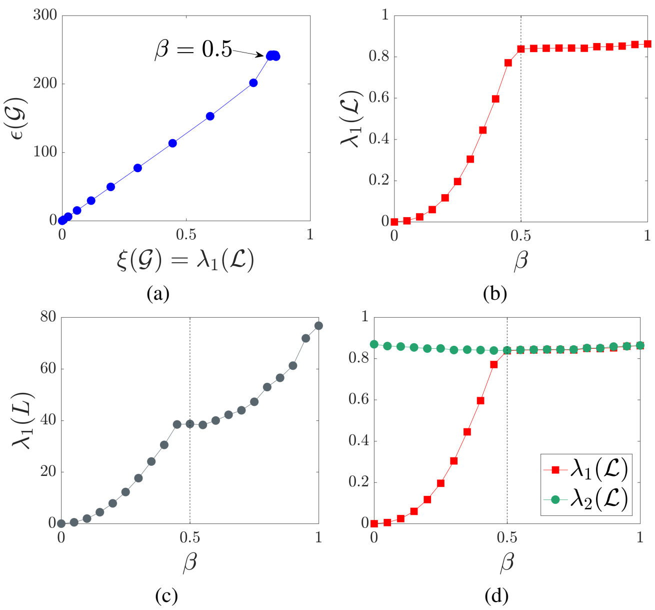

Fig. 1: Example 1. (a): Comparison between algebraic conflict $\xi(\mathcal{G})$ and frustration index $\epsilon(\mathcal{G})$. (b): $\lambda_1(\mathcal{L})$ and (c): $\lambda_1(L)$ as β increases. (d) Gap between $\lambda_1(\mathcal{L})$ and $\lambda_2(\mathcal{L})$ as β increases.

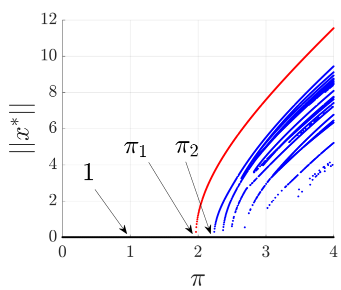

Fig. 2: Example 2. Norm of the equilibria x^* of the system (3), for different values of π. The first bifurcation happens at $\pi = \pi_1 > 1$.

VI. Conclusions

In this work we have investigated how the frustration of a social network influences the appearance of nonzero equilibria as function of a scalar parameter playing the role of social effort, for a particular class of nonlinear interconnected systems with sigmoidal, monotonically increasing and saturated nonlinearities, and with a signed adjacency graph.

We have shown that, similarly to the unsigned graph case, the system undergoes a pitchfork bifurcation beyond which two nontrivial equilibria appear. However, for a signed graph, the value of social effort at which the bifurcation happens is a function of the frustration of the graph, meaning that a higher level of frustration requires a higher value of social commitment to be able to achieve a nontrivial decision state.

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